

SAMELSON PRODUCTS IN p -REGULAR EXCEPTIONAL LIE GROUPS

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ABSTRACT. The (non)triviality of Samelson products of the inclusions of the spheres into p -regular exceptional Lie groups is completely determined, where a connected Lie group is called p -regular if it has the p -local homotopy type of a product of spheres.

1. INTRODUCTION AND STATEMENT OF THE RESULT

For a homotopy associative H-space with inverse X , the correspondence $X \wedge X \rightarrow X$, $(x, y) \mapsto xyx^{-1}y^{-1}$ induces a binary operation

$$\langle -, - \rangle : \pi_i(X) \otimes \pi_j(X) \rightarrow \pi_{i+j}(X)$$

called the Samelson product in X . We consider the basic Samelson products in p -regular Lie groups. Let G be a compact simply connected Lie group. By the Hopf theorem, G has the rational homotopy type of the product $S^{2n_1-1} \times \cdots \times S^{2n_\ell-1}$, where $n_1 \leq \cdots \leq n_\ell$. The sequence n_1, \dots, n_ℓ is called the type of G and is denoted by $\mathbf{t}(G)$. We here list the types of exceptional Lie groups.

| G | $\mathbf{t}(G)$ | G | $\mathbf{t}(G)$ |
|-------|-----------------|-------|------------------------------|
| G_2 | 2, 6 | E_6 | 2, 5, 6, 8, 9, 12 |
| F_4 | 2, 6, 8, 12 | E_7 | 2, 6, 8, 10, 12, 14, 18 |
| | | E_8 | 2, 8, 12, 14, 18, 20, 24, 30 |

We say that G is p -regular if it has the p -local homotopy type of a product of spheres. By the classical result of Serre, it is known that G is p -regular if and only if $p \geq n_\ell$, in which case

$$G_{(p)} \simeq S_{(p)}^{2n_1-1} \times \cdots \times S_{(p)}^{2n_\ell-1}.$$

Suppose that G is p -regular, and let ϵ_{2n_i-1} be the composite

$$S^{2n_i-1} \xrightarrow{\text{incl}} S_{(p)}^{2n_1-1} \times \cdots \times S_{(p)}^{2n_\ell-1} \simeq G_{(p)}$$

where if there are more than one i in $\mathbf{t}(G)$, we distinguish corresponding ϵ_{2i-1} but not write it explicitly. The Samelson products $\langle \epsilon_{2i-1}, \epsilon_{2j-1} \rangle$ are fundamental in studying the homotopy (non)commutativity of $G_{(p)}$ as in [KK] and its applications (See [KKTh, KKTs, Th], for example). So we would like to determine their (non)triviality. In [B], Bott computes the Samelson products in the classical groups $U(n)$ and $Sp(n)$. Then by combining with the information of the

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p -primary component of the homotopy groups of spheres [To], the (non)triviality of the Samelson products $\langle \epsilon_{2i-1}, \epsilon_{2j-1} \rangle$ is completely determined when $G = \mathrm{SU}(n), \mathrm{Sp}(n), \mathrm{Spin}(2n+1)$, where $\mathrm{Sp}(n)_{(p)} \simeq \mathrm{Spin}(2n+1)_{(p)}$ as loop spaces by [F] since p is odd. For example, when $G = \mathrm{SU}(n)$ and $p \geq n$, the type of G is given by $2, \dots, n$ and

$$\langle \epsilon_{2i-1}, \epsilon_{2j-1} \rangle \neq 0 \quad \text{if and only if} \quad i + j > p.$$

So apart from $\mathrm{Spin}(2n)$, all we have to consider is the exceptional Lie groups. The (non)triviality of the Samelson products $\langle \epsilon_{2i-1}, \epsilon_{2j-1} \rangle$ is known only in a few cases, and the most general result so far is:

Theorem 1.1 (Hamanaka and Kono [HK]). *Let G be a p -regular exceptional Lie group. If $i, j \in \mathfrak{t}(G)$ satisfy $i + j = p + 1$, then $\langle \epsilon_{2i-1}, \epsilon_{2j-1} \rangle$ is nontrivial.*

Remark 1.2. The Samelson products in G_2 are first computed in [O], and some more Samelson products in E_7 and E_8 are computed in [KK].

Based on this result, Kono posed the following conjecture (in a private communication).

Conjecture 1.3. Let G be a p -regular exceptional Lie group. For $i, j \in \mathfrak{t}(G)$, there exists $k \in \mathfrak{t}(G)$ satisfying $i + j = k + p - 1$ if and only if $\langle \epsilon_{2i-1}, \epsilon_{2j-1} \rangle$ is nontrivial.

Notice that the only if part of the conjecture follows immediately from the information of the p -primary component of the homotopy groups of spheres [To] (cf. [KK]). We will prove the if part and obtain:

Theorem 1.4. *Conjecture 1.3 is true.*

The paper is structured as follows. In §2, we reduce the nontriviality of the Samelson products $\langle \epsilon_{2i-1}, \epsilon_{2j-1} \rangle$ in the p -regular Lie group G to a certain condition of the Steenrod operation \mathcal{P}^1 on the mod p cohomology of the classifying space BG . Then for a p -regular exceptional Lie group G , we compute the mod p cohomology of BG as the ring of invariants of the Weyl group of G . With this description of the mod p cohomology of BG , we compute the action of \mathcal{P}^1 on it. In §3, we prove that the above condition on \mathcal{P}^1 is satisfied to complete the proof of Theorem 1.4.

2. MOD p COHOMOLOGY OF BG

2.1. Reduction. Let G be a compact simply connected Lie group. We first reduce Theorem 1.4 to the action of the Steenrod operation \mathcal{P}^1 on the mod p cohomology of the classifying space BG as in [HK, KK]. Recall that if the integral homology of G has no p -torsion, the mod p cohomology of the classifying space BG is given by

$$(2.1) \quad H^*(BG; \mathbb{Z}/p) = \mathbb{Z}/p[x_{2i} \mid i \in \mathfrak{t}(G)], \quad |x_j| = j.$$

When there are more than one i in $\mathfrak{t}(G)$, we distinguish corresponding x_{2i} but not write it explicitly as in the case of ϵ_{2i-1} in the preceding section.

Lemma 2.1. *Suppose that G is p -regular. For $i, j \in \mathfrak{t}(G)$, if there is $k \in \mathfrak{t}(G)$ such that $\mathcal{P}^1 x_{2k}$ involves $\lambda x_{2i} x_{2j}$ with $\lambda \neq 0$, then $\langle \epsilon_{2i-1}, \epsilon_{2j-1} \rangle$ is nontrivial.*

Proof. Let $\bar{\epsilon}_{2i} : S^{2i} \rightarrow BG_{(p)}$ be the adjoint of ϵ_{2i-1} for $i \in \mathfrak{t}(G)$, and so we may assume that $\bar{\epsilon}_{2i}^*(x_{2i}) = u_{2i}$ for a generator u_{2i} of $H^{2i}(S^{2i}; \mathbb{Z}/p)$. Assume that the Samelson product $\langle \epsilon_{2i-1}, \epsilon_{2j-1} \rangle$ is trivial, which is equivalent to the triviality of the Whitehead product $[\bar{\epsilon}_{2i}, \bar{\epsilon}_{2j}]$ by the adjointness of Samelson products and Whitehead products. Then the map $\bar{\epsilon}_{2i} \vee \bar{\epsilon}_{2j} : S^{2i} \vee S^{2j} \rightarrow BG_{(p)}$ extends to a map $\mu : S^{2i} \times S^{2j} \rightarrow BG_{(p)}$, up to homotopy. Hence since $\mathcal{P}^1 x_{2k}$ involves $\lambda x_{2i} x_{2j}$ with $\lambda \neq 0$, we have

$$\mu^*(\mathcal{P}^1 x_{2k}) = \mu^*(\lambda x_{2i} x_{2j}) = \lambda u_{2i} \times u_{2j} \neq 0.$$

On the other hand, by the naturality of \mathcal{P}^1 , we also have

$$\mu^*(\mathcal{P}^1 x_{2k}) = \mathcal{P}^1 \mu^*(x_{2k}) = 0$$

since \mathcal{P}^1 is trivial on $H^*(S^{2i} \times S^{2j}; \mathbb{Z}/p)$, which is a contradiction. Therefore the proof is completed. \square

By Lemma 2.1, we obtain the if part of Theorem 1.4 by the following.

Theorem 2.2. *Let G be a p -regular exceptional Lie group. If $i, j, k \in \mathfrak{t}(G)$ satisfy $i + j = k + p - 1$, $\mathcal{P}^1 x_{2k}$ involves $\lambda x_{2i} x_{2j}$ with $\lambda \neq 0$.*

The rest of this paper is devoted to prove Theorem 2.2.

2.2. Generators. In this subsection, we choose generators of the mod p cohomology of BG . We set notation. Hereafter, let p be a prime greater than 5. Recall that the integral homology of G is p -torsion free for $p > 5$, and so the mod p cohomology of BG is given as (2.1). For a homomorphism $\rho : H \rightarrow K$ between Lie groups, we denote the induced map $BH \rightarrow BK$ ambiguously by ρ .

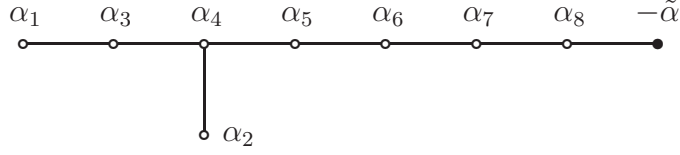
We first choose generators of the mod p cohomology of BE_8 . Let T be a maximal torus of E_8 . Then as in [MT], since $p > 5$, the inclusion $T \rightarrow E_8$ induces an isomorphism

$$(2.2) \quad H^*(BE_8; \mathbb{Z}/p) \xrightarrow{\cong} H^*(BT; \mathbb{Z}/p)^{W(E_8)},$$

where the right hand side is the ring of invariants of the Weyl group $W(E_8)$. We calculate invariants of $W(E_8)$ through a maximal rank subgroup of E_8 . Let $\epsilon_1, \dots, \epsilon_8$ be the standard basis of \mathbb{R}^8 which is regarded as the Lie algebra of T . As in [MT], we choose simple roots of E_8 as

$$\alpha_1 = \frac{1}{2}(\epsilon_1 + \epsilon_8) - \frac{1}{2}(\epsilon_2 + \epsilon_3 + \epsilon_4 + \epsilon_5 + \epsilon_6 + \epsilon_7), \quad \alpha_2 = \epsilon_1 + \epsilon_2, \quad \alpha_i = \epsilon_{i-1} - \epsilon_{i-2} \quad (3 \leq i \leq 8),$$

by which the extended Dynkin diagram of E_8 is described as



where $\tilde{\alpha}$ is the dominant root. Removing α_1 from the diagram, we get the maximal rank subgroup of E_8 which is of type D_8 . Then there is a homomorphism $\rho_1 : \text{Spin}(16) \rightarrow E_8$ which induces a monomorphism

$$\rho_1^* : H^*(BE_8; \mathbb{Z}/p) \rightarrow H^*(B\text{Spin}(16); \mathbb{Z}/p).$$

By putting $t_1 = -\epsilon_1$, $t_8 = -\epsilon_8$ and $t_i = \epsilon_i$ ($2 \leq i \leq 7$), $H^*(BT; \mathbb{Z}/p)$ is identified with the polynomial ring $\mathbb{Z}/p[t_1, \dots, t_8]$. Let c_i and p_i be the i -th elementary symmetric functions in t_1, \dots, t_8 and in t_1^2, \dots, t_8^2 , respectively. As in (2.2), we have an isomorphism

$$H^*(B\text{Spin}(16); \mathbb{Z}/p) \xrightarrow{\cong} \mathbb{Z}[t_1, \dots, t_8]^{W(D_8)} = \mathbb{Z}/p[p_1, \dots, p_7, c_8],$$

and then since $W(E_8)$ is generated by $W(D_8)$ and the reflection φ corresponding to the simple root α_1 , it follows from (2.2) that

$$H^*(BE_8; \mathbb{Z}/p) \cong \mathbb{Z}/p[p_1, \dots, p_7, c_8] \cap \mathbb{Z}/p[t_1, \dots, t_8]^\varphi.$$

In [HK], the action of φ on $p_1, \dots, p_8, c_8 \in \mathbb{Z}/p[t_1, \dots, t_8]$ is described as

$$\varphi(p_1) = p_1, \quad \varphi(p_i) \equiv p_i + h_i c_1, \quad \varphi(c_8) \equiv c_8 - \frac{1}{4} c_7 c_1 \pmod{(c_1^2)}$$

for $2 \leq i \leq 8$, where

$$\begin{aligned} h_2 &= \frac{3}{2} c_3, & h_3 &= -\frac{1}{2} (5c_5 + c_3 c_2), & h_4 &= \frac{1}{2} (7c_7 + 3c_5 c_2 - c_4 c_3), \\ h_5 &= -\frac{1}{2} (5c_7 c_2 - 3c_6 c_3 + c_5 c_4), & h_6 &= -\frac{1}{2} (5c_8 c_3 - 3c_7 c_4 + c_6 c_5), & h_7 &= \frac{1}{2} (3c_8 c_5 - c_7 c_6). \end{aligned}$$

We put

$$\begin{aligned} \hat{x}_4 &= p_1, \\ \hat{x}_{16} &= 12p_4 - \frac{18}{5} p_3 p_1 + p_2^2 + \frac{1}{10} p_2 p_1^2 + 168c_8, \\ \hat{x}_{24} &= 60p_6 - 5p_5 p_1 - 5p_4 p_2 + 3p_3^2 - p_3 p_2 p_1 + \frac{5}{36} p_2^3 + 110c_8 p_2, \\ \hat{x}_{28} &= 480p_7 + 40p_5 p_2 - 12p_4 p_3 - p_3 p_2^2 - 3p_4 p_2 p_1 + \frac{24}{5} p_3^2 p_1 + \frac{11}{36} p_2^3 p_1 + 312c_8 p_3 - 82c_8 p_2 p_1, \end{aligned}$$

$$\begin{aligned}
 \hat{x}_{36} = & 480p_7p_2 + 72p_6p_3 - 30p_5p_4 - \frac{25}{2}p_5p_2^2 + 9p_4p_3p_2 - \frac{18}{5}p_3^3 - \frac{1}{4}p_3p_2^3 + 1020c_8p_5 + 102c_8p_3p_2 \\
 & - 42p_6p_2p_1 + 9p_5p_3p_1 - \frac{3}{2}p_4p_2^2p_1 + \frac{9}{5}p_3^2p_2p_1 + \frac{1}{24}p_2^4p_1 - 330c_8p_4p_1 - \frac{89}{2}c_8p_2^2p_1 - 300c_8^2p_1 \\
 & + \frac{89}{4}p_5p_2p_1^2 - \frac{15}{2}p_4p_3p_1^2 - \frac{11}{20}p_3p_2^2p_1^2 + 156c_8p_3p_1^2 + \frac{5}{16}p_4p_2p_1^3 + \frac{9}{8}p_3^2p_1^3 + \frac{27}{320}p_2^3p_1^3 \\
 & - \frac{323}{8}c_8p_2p_1^3 - \frac{195}{32}p_5p_1^4 - \frac{13}{64}p_3p_2p_1^4 - \frac{7}{192}p_2^2p_1^5 + \frac{195}{32}c_8p_1^5 + \frac{3}{32}p_3p_1^6 - \frac{1}{1024}p_2p_1^7, \\
 \hat{x}_{40} = & 480p_7p_3 + 50p_6p_2^2 + 50p_5^2 - 10p_5p_3p_2 - \frac{25}{2}p_4^2p_2 + 9p_4p_3^2 - \frac{25}{36}p_4p_2^3 + \frac{3}{4}p_3^2p_2^2 + \frac{25}{864}p_2^5 \\
 & + 2400c_8p_6 + 250c_8p_4p_2 + 3550c_8^2p_2 + 6c_8p_3^2 - \frac{175}{18}c_8p_2^3, \\
 \hat{x}_{48} = & -200p_7p_5 - 60p_7p_3p_2 + 3p_6p_3^2 + \frac{25}{9}p_6p_3^3 + \frac{25}{3}p_5^2p_2 - \frac{5}{2}p_5p_4p_3 - \frac{25}{24}p_5p_3p_2^2 - \frac{25}{48}p_4^2p_2^2 \\
 & + p_4p_3^2p_2 + \frac{25}{864}p_4p_2^4 - \frac{3}{10}p_3^4 - \frac{1}{36}p_3^2p_2^3 - \frac{25}{62208}p_2^6 - 400c_8p_6p_2 - 115c_8p_5p_3 - \frac{25}{12}c_8p_4p_2^2 \\
 & + 3c_8p_3^2p_2 + \frac{25}{27}c_8p_2^4 + 75c_8p_4^2 - 300c_8^2p_4 - \frac{1525}{12}c_8^2p_2^2 + 300c_8^3.
 \end{aligned}$$

Hamanaka and Kono [HK] calculates φ -invariants in dimension 4, 16 and 24 as follows.

Proposition 2.3 (Hamanaka and Kono [HK]). *Let $\bar{x}_i \in \mathbb{Z}/p[p_1, \dots, p_7, c_8]$ with $|\bar{x}_i| = i$.*

(1) *If $\varphi(\bar{x}_i) \equiv \bar{x}_i \pmod{(c_1^2)}$ in $\mathbb{Z}/p[t_1, \dots, t_8]$ for $i = 4, 16$, then*

$$\bar{x}_4 = \alpha \hat{x}_4 \quad \text{and} \quad \bar{x}_{16} = \beta \hat{x}_{16} + \gamma \hat{x}_4^4 \quad (\alpha, \beta, \gamma \in \mathbb{Z}/p).$$

(2) *If $\varphi(\bar{x}_{24}) \equiv \bar{x}_{24} \pmod{(c_1^2, c_2^2)}$ in $\mathbb{Z}/p[t_1, \dots, t_8]$, then*

$$\bar{x}_{24} \equiv \alpha \hat{x}_{24} \quad (\alpha \in \mathbb{Z}/p).$$

We further calculate φ -invariants in dimension 28, 36, 40, 48, where a partial calculation in dimension 28 is given in [KK].

Proposition 2.4 (cf. [KK]). *Let $\bar{x}_i \in \mathbb{Z}/p[p_1, \dots, p_7, c_8]$ with $|\bar{x}_i| = i$.*

(1) *If $\varphi(\bar{x}_{28}) \equiv \bar{x}_{28} \pmod{(c_1^2, c_2^2)}$ in $\mathbb{Z}/p[t_1, \dots, t_8]$, then*

$$\bar{x}_{28} \equiv \alpha \hat{x}_{28} + \beta \hat{x}_4 \hat{x}_{24} \pmod{(p_1^2)} \quad (\alpha, \beta \in \mathbb{Z}/p).$$

(2) *If $\varphi(\bar{x}_{36}) \equiv \bar{x}_{36} \pmod{(c_1^2)}$ in $\mathbb{Z}/p[t_1, \dots, t_8]$, then*

$$\bar{x}_{36} = \alpha_1 \hat{x}_{36} + \alpha_2 \hat{x}_4 \hat{x}_{16}^2 + \alpha_3 \hat{x}_4^2 \hat{x}_{28} + \alpha_4 \hat{x}_4^3 \hat{x}_{24} + \alpha_5 \hat{x}_4^5 \hat{x}_{16} + \alpha_6 \hat{x}_4^9 \quad (\alpha_i \in \mathbb{Z}/p).$$

(3) *If $\varphi(\bar{x}_i) \equiv \bar{x}_i \pmod{(c_1^2, c_2)}$ in $\mathbb{Z}/p[t_1, \dots, t_8]$ for $i = 40, 48$, then*

$$\bar{x}_{40} \equiv \alpha_1 \hat{x}_{40} + \alpha_2 \hat{x}_{24} \hat{x}_{16}, \quad \bar{x}_{48} \equiv \beta_1 \hat{x}_{48} + \beta_2 \hat{x}_{24}^2 + \beta_3 \hat{x}_{16}^3 \pmod{(p_1)} \quad (\alpha_i, \beta_i \in \mathbb{Z}/p).$$

Proof. The proof is the same as Proposition 2.3 given in [HK], and we only consider \bar{x}_{28} since other cases are analogous. Excluding the indeterminacy $\hat{x}_4\hat{x}_{24}$, we may suppose that \bar{x}_{28} is a linear combination

$$\lambda_1 p_7 + \lambda_2 p_5 p_2 + \lambda_3 p_4 p_3 + \lambda_4 p_4 p_2 p_1 + \lambda_5 p_3^2 p_1 + \lambda_6 p_3 p_2^2 + \lambda_7 p_2^3 p_1 + \lambda_8 c_8 p_3 + \lambda_9 c_8 p_2 p_1$$

for $\lambda_i \in \mathbb{Z}/p$. By the congruence $\varphi(\bar{x}_{28}) \equiv \bar{x}_{28} \pmod{(c_1^2, c_2^2)}$ and the equality $p_i = \sum_{j+k=2i} (-1)^{i+j} c_j c_k$, we get linear equations in $\lambda_1, \dots, \lambda_9$. Solving these equations, we see that $\bar{x}_{28} \equiv \alpha \hat{x}_{28} \pmod{(c_1^2, c_2^2)}$, thus the proof is completed since the intersection of the ideal (c_1^2, c_2^2) and the subring $\mathbb{Z}/p[p_1, \dots, p_7, c_8]$ of $\mathbb{Z}/p[t_1, \dots, t_8]$ is the ideal (p_1^2) in $\mathbb{Z}/p[p_1, \dots, p_7, c_8]$. \square

As an immediate consequence of Proposition 2.3 and 2.4, we obtain:

Corollary 2.5. *We can choose a generator x_i of $H^*(BE_8; \mathbb{Z}/p)$ for $i \neq 60$ in such a way that*

$$\begin{aligned} \rho_1^*(x_i) &= \hat{x}_i & (i = 4, 16, 36), & \quad \rho_1^*(x_i) \equiv \hat{x}_i \pmod{(p_1^2)} \quad (i = 24, 28) \\ \rho_1^*(x_i) &\equiv \hat{x}_i \pmod{(p_1)} & (i = 40, 48). \end{aligned}$$

Hereafter, we choose generators of $H^*(BE_8, \mathbb{Z}/p)$ as in Corollary 2.5. From these generators, we next choose generators of $H^*(BG; \mathbb{Z}/p)$ for $G = F_4, E_6, E_7$. Recall that there is a commutative diagram of the canonical homomorphisms

$$(2.3) \quad \begin{array}{ccccccc} F_4 & \xrightarrow{\alpha_3} & E_6 & \xrightarrow{\alpha_2} & E_7 & \xrightarrow{\alpha_1} & E_8 \\ \uparrow \rho_4 & & \uparrow \rho_3 & & \uparrow \rho_2 & & \uparrow \rho_1 \\ \text{Spin}(9) & \xrightarrow{\theta_3} & \text{Spin}(10) & \xrightarrow{\theta_2} & \text{Spin}(12) & \xrightarrow{\theta_1} & \text{Spin}(16). \end{array}$$

Let us consider the induced map of the arrows in the mod p cohomology of the classifying spaces. Obviously, we have

$$(2.4) \quad \theta_1^*(p_i) = p_i \ (i = 1, 2, 3, 4, 5), \quad \theta_1^*(p_6) = c_6^2, \quad \theta_1^*(p_7) = 0, \quad \theta_1^*(c_8) = 0,$$

$$(2.5) \quad \theta_2^*(p_i) = p_i \ (i = 1, 2, 3, 4), \quad \theta_2^*(p_5) = c_5^2, \quad \theta_2^*(c_6) = 0,$$

$$(2.6) \quad \theta_3^*(p_i) = p_i \ (i = 1, 2, 3, 4), \quad \theta_3^*(c_5) = 0.$$

To determine the induced map of α_i , we recall the results of [A, C, N, TW, W].

Proposition 2.6. (1) $H^*(E_6/\text{Spin}(10); \mathbb{Z}/p) = \mathbb{Z}/p[y_8]/(y_8^3) \otimes \Lambda(y_{17})$, $|y_i| = i$.
 (2) $H^*(E_6/F_4; \mathbb{Z}/p) = \Lambda(z_9, z_{17})$, $|z_i| = i$.
 (3) $\tilde{H}^*(E_7/E_6; \mathbb{Z}/p) = \mathbb{Z}/p\langle z_{10}, z_{18} \rangle$, $|z_i| = i$ for $* < 37$.
 (4) $H^*(E_8/E_7; \mathbb{Z}/p) = \mathbb{Z}/p[z_{12}, z_{20}]$, $|z_i| = i$ for $* < 40$.

We next choose generators of $H^*(BG; \mathbb{Z}/p)$ for $G \neq E_8$. Let

$$\hat{x}_{10} = c_5, \quad \hat{x}_{12} = -6p_3 + p_2 p_1 - 60c_6, \quad \hat{x}_{18} = p_2 c_5 \quad \text{and} \quad \hat{x}_{20} = p_5 + p_2 c_6.$$

We abbreviate $\theta_i(\hat{x}_j)$ by \hat{x}_j .

Corollary 2.7. *We can choose a generator x_i of $H^*(BE_7; \mathbb{Z}/p)$ as*

$$\rho_2^*(x_i) = \hat{x}_i \quad (i = 4, 12, 16, 36) \quad \text{and} \quad \rho_2^*(x_i) \equiv \hat{x}_i \pmod{(p_1^2)} \quad (i = 20, 24, 28).$$

Proof. Consider the Serre spectral sequence of the homotopy fiber sequence $E_8/E_7 \rightarrow BE_7 \rightarrow BE_8$. Then by Proposition 2.6, we get $\alpha_1^*(x_i) = x_i$ for $i = 4, 16, 24, 28, 36$, hence the desired result for $\rho_2^*(x_i)$ by Corollary 2.5. As in [BH], we can choose a generator x_{12} of $H^*(BF_4; \mathbb{Z}/p)$ as $\rho_4^*(x_{12}) = -6p_3 + p_2p_1$. On the other hand, it is calculated in [N] that $\rho_2^*(x_{12}) \equiv -6p_3 - 60c_6$ modulo decomposables. Then we get $\rho_2^*(x_{12}) = \hat{x}_{12}$ by (2.5) and (2.6). By the Serre spectral sequence of the homotopy fiber sequence $E_6/\text{Spin}(10) \rightarrow B\text{Spin}(10) \rightarrow BE_6$ and Proposition 2.6, we have $\rho_3^*(x_{10}) \neq 0$. Then for a degree reason, we may choose $x_{10} \in H^*(BE_6; \mathbb{Z}/p)$ as $\rho_3^*(x_{10}) = c_5$. Consider next the Serre spectral sequence of the homotopy fiber sequence $E_7/E_6 \rightarrow BE_6 \rightarrow BE_7$. Then it follows from Proposition 2.6 that we may choose $x_{20} \in H^*(BE_7; \mathbb{Z}/p)$ as $\alpha_2^*(x_{20}) = x_{10}^2$, hence $\rho_2^*(x_{20}) \equiv p_5 + \alpha p_2c_6 \pmod{(p_1^2)}$ by (2.5), where $\alpha \in \mathbb{Z}/p$. For a degree reason, we have $\alpha_1^*(x_{40}) \equiv \lambda x_{20}^2 \pmod{(x_4, x_{12}, x_{16})}$, hence

$$\theta_2^*(\hat{x}_{40}) = \lambda(p_5 + \alpha p_2c_6)^2 \pmod{(\hat{x}_4, \hat{x}_{12}, \hat{x}_{16})}.$$

Since $\theta_2^*(\hat{x}_{40}) \equiv 50p_5^2 - 10p_5p_3p_2 + \frac{1}{2}p_3^2p_2^2$ and $\hat{x}_{20}^2 \equiv p_5^2 - \frac{\alpha}{5}p_5p_3p_2 + \frac{\alpha^2}{100}p_3^2p_2^2 \pmod{(\hat{x}_4, \hat{x}_{12}, \hat{x}_{16})}$, we get $\alpha = 1$ and $\lambda = 50$ \square

Corollary 2.8. *We can choose a generator x_i of $H^*(BE_6; \mathbb{Z}/p)$ as*

$$\rho_3^*(x_i) = \hat{x}_i \quad (i = 4, 12, 10, 16, 18) \quad \text{and} \quad \rho_3^*(x_{24}) = \hat{x}_{24} \pmod{(p_1^2)}.$$

Proof. By the Serre spectral sequence of the homotopy fiber sequence $E_7/E_6 \rightarrow BE_6 \rightarrow BE_7$ together with Proposition 2.6 and Corollary 2.7, we get $\alpha_2^*(x_i) = x_i$ for $i = 4, 12, 16, 24$. Then we obtain the desired result for x_i ($i = 4, 12, 16, 24$) by Corollary 2.7. The second equality holds for a degree reason as mentioned in the proof of Corollary 2.7. We see that $\rho_3^*(x_{28})$ involves the term $p_2c_5^2$, implying $\rho_3^*(x_{18}) \neq 0$. Then for a degree reason, we may choose x_{18} as desired. \square

Corollary 2.9. *We can choose a generator x_i of $H^*(BF_4; \mathbb{Z}/p)$ as*

$$\rho_4^*(x_i) = \hat{x}_i \quad (i = 4, 12, 16) \quad \text{and} \quad \rho_4^*(x_{24}) \equiv \hat{x}_{24} \pmod{(p_1^2)}.$$

Proof. The result follows from the Serre spectral sequence of the homotopy fiber sequence $E_6/F_4 \rightarrow BF_4 \rightarrow BE_6$ together with Proposition 2.6 and Corollary 2.8. \square

Recall that G_2 is a subgroup of $\text{Spin}(7)$. We denote the inclusion $G_2 \rightarrow \text{Spin}(7)$ by ρ .

Proposition 2.10. *The induced map of $\rho : BG_2 \rightarrow B\text{Spin}(7)$ in the mod p cohomology satisfies*

$$\rho^*(p_1) = x_4, \quad \rho^*(p_2) = 0 \quad \text{and} \quad \rho^*(p_3) = x_{12}.$$

Proof. It is well known that $\text{Spin}(7)/G_2 = S^7$. Then by considering the Serre spectral sequence of the homotopy fiber sequence $\text{Spin}(7)/G_2 \rightarrow BG_2 \rightarrow B\text{Spin}(7)$, we obtain the desired result. \square

For the rest of this paper, we choose generators of $H^*(BG; \mathbb{Z}/p)$ as in Corollary 2.7, 2.8, 2.9, 2.10.

2.3. Calculation of $\mathcal{P}^1 \rho_i^*(x_j)$. Let s_k denote the power sum $t_1^{2k} + \cdots + t_m^{2k}$. The power sum s_k can be expanded in terms of p_1, \dots, p_m by the Girard's formula

$$(2.7) \quad s_k = (-1)^k k \sum_{i_1+2i_2+\cdots+mi_m=k} (-1)^{i_1+\cdots+i_m} \frac{(i_1+\cdots+i_m-1)!}{i_1! \cdots i_m!} p_1^{i_1} \cdots p_m^{i_m}.$$

By the following lemma together with Girard's formula, we can determine $\mathcal{P}^1 p_n$ and $\mathcal{P}^1 c_m$.

Lemma 2.11. *For $q = \frac{p-1}{2}$, there holds*

$$\mathcal{P}^1 p_n = 2 \sum_{i=0}^{q-1} (-1)^i p_{n+i} s_{q-i} + 2(-1)^q (n+q) p_{n+q} \quad \text{and} \quad \mathcal{P}^1 c_m = c_m s_q.$$

Proof. For $1 \leq i_1 < \cdots < i_k \leq m$, we have

$$\begin{aligned} (t_{i_1}^{2m} + \cdots + t_{i_k}^{2m}) t_{i_1}^2 \cdots t_{i_k}^2 &= s_m t_{i_1}^2 \cdots t_{i_k}^2 - s_{m-1} \sum_{i_0 \neq i_1, \dots, i_k} t_{i_0}^2 t_{i_1}^2 \cdots t_{i_k}^2 \\ &\quad + \sum_{i_0 \neq i_1, \dots, i_k} (t_{i_0}^{2m-2} + t_{i_1}^{2m-2} + \cdots + t_{i_k}^{2m-2}) t_{i_0}^2 t_{i_1}^2 \cdots t_{i_k}^2. \end{aligned}$$

Then since $\mathcal{P}^1 p_n = 2 \sum_{1 \leq i_1 < \cdots < i_n \leq m} (t_{i_1}^{2q} + \cdots + t_{i_n}^{2q}) t_{i_1}^2 \cdots t_{i_n}^2$, we obtain the first equality. The second equality is obvious. \square

Proposition 2.12. *Define ideals I_k of $\mathbb{Z}/p[p_1, \dots, p_7, c_8]$ for $k = 0, \dots, 8$ as*

$$\begin{aligned} I_0 &= (p_1, p_2^2, p_3^3, p_4^2, p_6^2, c_8), & I_1 &= I_0 + (p_3, p_6), & I_2 &= I_0 + (p_2, p_3^2, p_4, p_7^2), \\ I_3 &= I_0 + (p_2, p_3^2, p_6), & I_4 &= I_0 + (p_2, p_3^2, p_4), & I_5 &= I_0 + (p_2, p_3, p_4, p_6, p_7), \\ I_6 &= I_0 + (p_2, p_3^2, p_4, p_6), & I_7 &= I_0 + (p_2, p_3^2, p_4, p_6, p_7^2), & I_8 &= I_0 + (p_2, p_4, p_7^4, \hat{x}_{24}). \end{aligned}$$

Then for a generator $x_k \in H^*(BE_8; \mathbb{Z}/p)$, we have the following table.

| p | k | $\mathcal{P}^1 \rho_1^*(x_k) \bmod I$ | I | p | k | $\mathcal{P}^1 \rho_1^*(x_k) \bmod I$ | I |
|-----|-----|---|-------|-----|-----|---|-------|
| 31 | 16 | $9p_7^2 p_5 + 24p_7 p_5^2 p_2 + 22p_5^3 p_4$ | I_1 | 37 | 4 | $p_7^2 p_5 + 34p_7 p_5^2 p_2 + 36p_5^3 p_4$ | I_1 |
| | 24 | $28p_7 p_6 p_5 p_3 + 16p_6 p_5^3$ | I_2 | | 16 | $8p_7^2 p_5 p_3 + 27p_7 p_5^3 + 2p_5^3 p_4 p_3$ | I_3 |
| | 28 | $27p_7^2 p_5 p_3 + 30p_7 p_5^3 + 30p_5^3 p_4 p_3$ | I_3 | | 24 | $5p_7^3 p_3 + 27p_7^2 p_5^2 + 36p_6 p_5^3 p_3$ | I_4 |
| | 36 | $p_7^3 p_3 + 10p_7^2 p_5^2 + 6p_6 p_5^3 p_3$ | I_4 | | 28 | $7p_5^5$ | I_5 |
| | 40 | $8p_5^5$ | I_5 | | 36 | $20p_7^2 p_5^2 p_3 + 35p_7 p_5^4$ | I_6 |
| | 48 | $4p_7^2 p_5^2 p_3 + 5p_7 p_5^4$ | I_6 | | 48 | $36p_7 p_5^4 p_3 + 3p_5^6$ | I_7 |

| | | | | | | | |
|----|----|--|-------|----|----|---|-------|
| 41 | 4 | $35p_7p_6p_5p_3 + 40p_6p_5^3$ | I_2 | 43 | 4 | $3p_7^2p_5p_3 + p_7p_5^3 + 39p_5^3p_4p_3$ | I_3 |
| | 16 | $9p_7^3p_3 + 38p_7^2p_5^2 + 16p_6p_5^3p_3$ | I_4 | | 16 | $9p_5^5$ | I_5 |
| | 28 | $7p_7^2p_5^2p_3 + 6p_7p_5^4$ | I_6 | | 24 | $11p_7^2p_5^2p_3 + 40p_7p_5^4$ | I_6 |
| | 40 | $34p_7p_5^4p_3 + 16p_5^6$ | I_7 | | 36 | $35p_7p_5^4p_3 + 42p_5^6$ | I_7 |
| 47 | 4 | $p_7^3p_3 + 25p_7^2p_5^2 + 43p_6p_5^3p_3$ | I_4 | 53 | 4 | $6p_7^2p_5^2p_3 + p_7p_5^4$ | I_6 |
| | 16 | $35p_7^2p_5^2p_3 + 10p_7p_5^4$ | I_6 | | 16 | $23p_7p_5^4p_3 + 39p_5^6$ | I_7 |
| | 28 | $17p_7p_5^4p_3 + 23p_5^6$ | I_7 | 59 | 4 | $5p_7p_5^4p_3 + 10p_5^6$ | I_7 |

For $p = 31$, we also have

$$\mathcal{P}^1\rho_1^*(x_{48}) \equiv 17p_7^3p_3^2 + 4p_7^2p_5^2p_3 + 5p_7p_5^4, \quad \mathcal{P}^2\rho_1^*(x_{48}) \equiv 26p_7^3p_5^3p_3^2 + 5p_7^2p_5^5p_3 + 8p_7p_5^7 \pmod{I_8}.$$

Proof. For $i = 4, 16, 24, 28, 36$, we have $\rho_1^*(x_i) \equiv \hat{x}_i \pmod{(p_1^2)}$. Since $\mathcal{P}^1(p_1^2) \subset (p_1)$ by the Cartan formula, we have $\mathcal{P}^1\rho_1^*(x_i) \equiv \mathcal{P}^1\hat{x}_i \pmod{(p_1)}$. For $i = 40, 48$, we analogously have $\mathcal{P}^1\rho_1^*(x_i) = \mathcal{P}^1\hat{x}_i + (\mathcal{P}^1p_1)q$ for some polynomial q in p_2, \dots, p_7, c_8 . For a degree reason, we have $q \equiv 0 \pmod{(p_1, p_2, p_3^2, p_4, p_6, c_8)}$, implying that $\mathcal{P}^1\rho_1^*(x_i) \equiv \mathcal{P}^1\hat{x}_i \pmod{I}$ for the prescribed ideal I . Thus in order to fill the table, we only need to calculate $\mathcal{P}^1\hat{x}_i$ by Lemma 2.11 and Girard's formula (2.7).

For $p = 31$, we have $\mathcal{P}^1\rho_1^*(x_{48}) \equiv \mathcal{P}^1\hat{x}_{48} + (\mathcal{P}^1p_1)q \pmod{(p_1)}$ for some polynomial q in p_2, \dots, p_7, c_8 as above. Since $\hat{x}_i \in I_8$ for $i = 4, 16, 24, 36$, we have $\mathcal{P}^1p_1 \equiv 0 \pmod{I_8}$ for a degree reason, hence $\mathcal{P}_1\rho_1^*(x_{48}) \equiv \mathcal{P}^1\hat{x}_{48} \pmod{I_8}$. Then we can calculate $\mathcal{P}^1\rho_1^*(x_{48}) \pmod{I_8}$ by Lemma 2.11. Since $\mathcal{P}^2p_1 = p_1^p$ and $\rho_1^*(x_{48}) \equiv \hat{x}_{48} \pmod{(p_1)}$, we have $\mathcal{P}^2\rho_1^*(x_{48}) \equiv \mathcal{P}^2\hat{x}_{48} \pmod{(p_1)}$. Now $\mathcal{P}^2\rho_1(x_{48})$ for $p = 31$ can be calculated from Lemma 2.11 and Girard's formula (2.7) together with the Adem relation $\mathcal{P}^1\mathcal{P}^1 = 2\mathcal{P}^2$. \square

Quite similarly to Proposition 2.12, we can calculate $\mathcal{P}^1\rho_i^*(x_j)$ for $G = E_7, E_6$.

Proposition 2.13. *For a generator $x_k \in H^*(BE_7; \mathbb{Z}/p)$, we have the following table.*

| p | k | $\mathcal{P}^1\rho_2^*(x_k) \pmod{I}$ | I |
|-----|-----|--|--------------------------|
| 19 | 12 | $18p_5^2p_2 + 3p_5p_4p_3 + 15p_5p_3p_2^2 + 10p_4^3 + 17p_4^2p_2^2 + 6p_4p_2^4 + 15p_2^6$ | (p_1, p_3^2, c_6) |
| | 16 | $11p_5p_4^2 + 16p_5p_4p_2^2 + 15p_5p_2^4$ | (p_1, p_3, c_6) |
| | 20 | $p_5^2p_4 + 18p_5^2p_2^2 + 17p_5p_4p_3p_2 + p_5p_3p_2^3 + 4c_6p_5p_4p_2 + 12c_6p_5p_2^3$ $+ 16c_6p_4^2p_3 + 8p_4c_6p_3p_2^2 + 7c_6p_3p_2^4$ | (p_1, p_3^2, c_6^2) |
| | 24 | $13p_5p_4^2p_2 + 7p_5p_4p_2^3 + 8p_5p_2^5$ | (p_1, p_3, p_5^2, c_6) |
| | 28 | $14p_5^2p_4p_2 + p_5^2p_2^3 + 8p_5p_4^2p_3 + 10p_5p_4p_3p_2^2 + 17p_5p_3p_2^4 + p_4^4 + 9p_4^3p_2^2$ $+ 6p_4^2p_2^4 + p_4p_2^6 + 3p_2^8$ | (p_1, p_3^2, c_6^2) |
| | 36 | $9p_5^2p_4^2 + 4p_5^2p_4p_2^2 + 6p_5^2p_4^2 + 17p_5p_4^2p_3p_2 + 15p_5p_3p_2^5 + 4p_4^4p_2 + 5p_4^3p_2^3$ $+ 2p_4^2p_2^5 + 11p_4p_2^7 + 3p_2^9$ | (p_1, p_3^2, c_6^2) |

| | | | |
|----|----|---|--------------------------|
| 23 | 4 | $22p_5^2p_2 + 21p_5p_4p_3 + 3p_5p_3p_2^2 + 15p_4^3 + 13p_4^2p_2^2 + 22p_4p_2^4 + 4p_2^6$ | (p_1, p_3^2, c_6) |
| | 12 | $7p_5^2p_4 + 6p_5^2p_2^2 + 14p_5p_4p_3p_2 + 13p_5p_3p_2^3 + 10p_4^3p_2 + 18p_4^2p_2^3 + 21p_4p_2^5$ $+ 4p_2^7 + 14c_6p_5p_4p_2 + 16c_6p_5p_2^3 + 7c_6p_4^2p_3 + 2p_4c_6p_3p_2^2 + 7c_6p_3p_2^4$ | (p_1, p_3^2, c_6^2) |
| | 16 | $3p_5p_4^2p_2 + 20p_5p_4p_2^3 + 19p_5p_2^5$ | (p_1, p_3, p_5^2, c_6) |
| | 28 | $9p_5^2p_4^2 + 3p_5^2p_4p_2^2 + 2p_5^2p_2^4 + 10p_5p_4^2p_3p_2 + 10p_5p_4p_3p_2^3 + 8p_5p_3p_2^5$ $+ 14p_4^4p_2 + 15p_4^3p_2^3 + 14p_2^9 + 9p_4^2p_2^5 + 15p_4p_2^7$ | (p_1, p_3^2, c_6^2) |
| 29 | 4 | $26p_5p_4^2p_2 + 4p_5p_4p_2^3 + 28p_5p_2^5$ | (p_1, p_3, p_5^2, c_6) |
| | 16 | $19p_5^2p_4^2 + p_5^2p_4p_2^2 + 19p_5^2p_2^4 + 10p_5p_4^2p_3p_2 + 6p_5p_4p_3p_2^3 + 13p_5p_3p_2^5$ $+ p_4^4p_2 + 7p_4^3p_2^3 + 2p_4^2p_2^5 + 16p_4p_2^7 + 21p_2^9$ | (p_1, p_3^2, c_6^2) |
| 31 | 12 | $p_5^3p_3 + 17p_5^2p_4^2 + 10p_5^2p_4p_2^2 + 28p_5^2p_2^4 + 4p_5p_4^2p_3p_2 + 18p_5p_4p_3p_2^3$ $+ 21p_2p_4^4 + 3p_4^3p_2^3 + 6p_4p_2^7 + 4p_5^3p_2^9 + 10c_6p_5^3 + 3c_6p_5^2p_3p_2 + 3c_6p_5p_4^2p_2$ $+ 27c_6p_5p_4p_2^3 + c_6p_5p_2^5 + c_6p_4^3p_3 + 25c_6p_4^2p_3p_2^2 + 5c_6p_4p_3p_2^4 + 30c_6p_3p_2^6$ | (p_1, p_3^2, c_6^2) |

Proposition 2.14. *For a generator $x_k \in H^*(BE_6; \mathbb{Z}/p)$, we have the following table.*

| p | k | $\mathcal{P}^1 \rho_3^*(x_k) \mod I$ | I |
|-----|-----|---|------------------------|
| 13 | 10 | $6c_5p_4p_2 + 11c_5p_2^3$ | (p_1, p_3^2, c_5^2) |
| | 12 | $10p_4p_3p_2 + 12p_3p_2^3 + 4c_5^2p_4 + c_5^2p_2^2$ | (p_1, p_3^2) |
| | 16 | $5p_2^5$ | (p_1, p_3, p_4, c_5) |
| | 18 | $5c_5p_4^2 + 9c_5p_4p_2^2 + 7c_5p_2^4$ | (p_1, p_3, c_5^2) |
| | 24 | $p_4^3 + 4p_4^2p_2^2 + 12p_4p_2^4 + 7p_2^6$ | (p_1, p_3, c_5) |
| 17 | 4 | $2p_4p_3p_2 + 16p_3p_2^3 + 16c_5^2p_4 + c_5^2p_2^2$ | (p_1, p_3^2) |
| | 10 | $4c_5p_4^2 + 9c_5p_4p_2^2 + 2c_5p_2^4$ | (p_1, p_3, c_5^2) |
| | 16 | $11p_4^3 + p_4^2p_2^2 + 8p_4p_2^4 + 8p_2^6$ | (p_1, p_3, c_5) |

We finally calculate $\mathcal{P}^1 x_k$ for a generator $x_k \in H^*(BG_2; \mathbb{Z}/p)$.

Proposition 2.15. *For a generator $x_k \in H^*(BG_2; \mathbb{Z}/p)$, we have*

$$\mathcal{P}^1 x_k = \begin{cases} x_4 x_{12} + 2x_4^4 & (k, p) = (4, 7) \\ 6x_{12}^2 + 2x_4^3 x_{12} & (k, p) = (12, 7) \\ 6x_{12}^2 + x_4^3 x_{12} + 2x_4^6 & (k, p) = (4, 11). \end{cases}$$

Proof. By Proposition 2.10 and the naturality of \mathcal{P}^1 , we have $\mathcal{P}^1 x_{4k} = \mathcal{P}^1 \rho^*(p_k) = \rho^*(\mathcal{P}^1 p_k)$, hence the proof is completed by Lemma 2.11 and Girard's formula (2.7). \square

3. PROOF OF THEOREM 2.2

In this section, we prove Theorem 2.2 by summarizing results in the previous section.

3.1. **The case of E_8 .** Suppose that E_8 is p -regular, that is, $p > 30$. By an easy degree consideration, we see that if $\mathcal{P}^1 x_k \bmod (x_{2i} \mid i \in \mathfrak{t}(E_8))^3$ is nontrivial for a generator x_k of $H^*(BE_8; \mathbb{Z}/p)$, it is as in the following table.

| | $\mathcal{P}^1 x_k \bmod (x_{2i} \mid i \in \mathfrak{t}(E_8))^3$ | (k, p) |
|-----|--|--|
| (1) | $\lambda_1 x_4 x_{60} + \lambda_2 x_{16} x_{48} + \lambda_3 x_{24} x_{40} + \lambda_4 x_{28} x_{36}$ | $(4, 31)$ |
| (2) | $\lambda_1 x_{16} x_{60} + \lambda_2 x_{28} x_{48} + \lambda_3 x_{36} x_{40}$ | $(16, 31), (4, 37)$ |
| (3) | $\lambda_1 x_{24} x_{60} + \lambda_2 x_{36} x_{48}$ | $(24, 31), (4, 41)$ |
| (4) | $\lambda_1 x_{28} x_{60} + \lambda_2 x_{40} x_{48}$ | $(28, 31), (16, 37), (4, 43)$ |
| (5) | $\lambda_1 x_{36} x_{60} + \lambda_2 x_{48}^2$ | $(36, 31), (24, 37), (16, 41), (4, 47)$ |
| (6) | $\lambda x_{40} x_{60}$ | $(40, 31), (28, 37), (16, 43)$ |
| (7) | $\lambda x_{48} x_{60}$ | $(48, 31), (36, 37), (28, 41), (24, 43),$ $(16, 47), (4, 53)$ |
| (8) | λx_{60}^2 | $(60, 31), (48, 37), (40, 41), (36, 43),$ $(28, 47), (16, 53), (4, 59)$ |

Let I_k for $k = 1, \dots, 8$ be the ideals of $\mathbb{Z}/p[p_1, \dots, p_7, c_8]$ as in Proposition 2.12.

(1) It is proved in [HK] that $\lambda_i \neq 0$ for $i = 1, 2, 3, 4$.

(2) Since $\hat{x}_i \in I_1$ for $i = 4, 16, 24$, for a degree reason, we have

$$\rho_1^*(\mathcal{P}^1 x_k) \equiv \lambda_2 \hat{x}_{28} \hat{x}_{48} + \lambda_3 \hat{x}_{36} \hat{x}_{40} \equiv 24000(\lambda_2(4p_7^2 p_5 + p_7 p_5^2 p_2) + \lambda_3 p_7 p_5^2 p_2) \bmod I_1 + (p_4).$$

On the other hand, by the naturality of \mathcal{P}^1 and Proposition 2.12,

$$\rho_1^*(\mathcal{P}^1 x_k) = \mathcal{P}^1 \rho_1(x_k) \equiv \begin{cases} 24p_7 p_5^2 p_2 + 9p_7^2 p_5 & (p = 31) \\ 34p_7 p_5^2 p_2 + p_7^2 p_5 & (p = 37) \end{cases} \bmod I_1 + (p_4),$$

implying that $(\lambda_2, \lambda_3) = (10, 14), (-9, 6)$ according as $p = 31, 37$. Since $\hat{x}_4, \hat{x}_{16}^2, \hat{x}_{24}, \hat{x}_{36} \in I_1 + (p_2, p_7)$, we also have

$$\rho_1^*(\mathcal{P}^1 x_k) \equiv \lambda_1 \hat{x}_{16} \rho_1^*(x_{60}) + \lambda_3 \hat{x}_{36} \hat{x}_{40} \equiv \lambda_1 \hat{x}_{16} \rho_1^*(x_{60}) - 1500 \lambda_3 p_5^3 p_4 \bmod I_1 + (p_2, p_7),$$

and by Proposition 2.12,

$$\rho_1^*(\mathcal{P}^1 x_k) = \mathcal{P}^1 \rho_1^*(x_k) \equiv \begin{cases} 22p_5^3 p_4 & (p = 31) \\ 36p_5^3 p_4 & (p = 37) \end{cases} \bmod I_1 + (p_2, p_7),$$

implying that $\lambda_1 \neq 0$ for both $p = 31, 37$.

(3) Since $\hat{x}_i, \hat{x}_j^2 \in I_2$ for $i = 4, 16$ and $j = 24, 28, 36$, we have

$$\rho_1^*(\mathcal{P}^1 x_k) \equiv \lambda_1 \hat{x}_{24} \rho_1^*(x_{60}) + \lambda_2 \hat{x}_{36} \hat{x}_{48} \equiv \lambda_1 \hat{x}_{24} \rho_1^*(x_{60}) - 14400 \lambda_2 p_7 p_6 p_5 p_3 \bmod I_2.$$

By the naturality of \mathcal{P}^1 and Proposition 2.12, we also have

$$\rho_1^*(\mathcal{P}^1 x_k) = \mathcal{P}^1 \rho_1^*(x_k) \equiv \begin{cases} 28p_7 p_6 p_5 p_3 + 16p_6 p_5^3 & (p = 31) \\ 35p_7 p_6 p_5 p_3 + 40p_6 p_5^3 & (p = 41) \end{cases} \bmod I_2,$$

implying that $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$ for both $p = 31, 41$.

(4) Since $\hat{x}_i, \hat{x}_{28}^2 \in I_3 + (p_3, p_4, p_7^2, \hat{x}_{40})$ for $i = 4, 16, 24, 36, 40$, it follows from Proposition 2.12 that

$$\lambda_1 \hat{x}_{28} \rho_1^*(x_{60}) \equiv \rho_1^*(\mathcal{P}^1 x_k) \equiv \mathcal{P}^1 \rho_1^*(x_k) \not\equiv 0 \pmod{I_3 + (p_3, p_4, p_7^2, \hat{x}_{40})}$$

so $\lambda_1 \neq 0$. We can similarly get $\lambda_2 \neq 0$ by considering $\rho_1^*(\mathcal{P}^1 x_k) \pmod{I_3 + (p_7^2, \hat{x}_{28})}$ since $\hat{x}_i \in I_3 + (p_7^2, \hat{x}_{28})$ for $i = 4, 16, 24, 28$.

(5), (6) and (7) We get $\lambda \neq 0$ similarly to (4) by considering $\rho_1^*(\mathcal{P}^1 x_k)$ modulo the ideals $I_4 + (p_7), I_5, I_6 + (\hat{x}_{40}^2)$ respectively for (5), (6) and (7) since $\hat{x}_4, \hat{x}_{16}, \hat{x}_{24}^2, \hat{x}_{36}^2 \in I_4 + (p_7), \hat{x}_i \in I_5$ for $i = 4, 16, 24, 18, 36$ and $\hat{x}_i \in I_6 + (\hat{x}_{40}^2)$ for $i = 4, 16, 24, 36, 40$.

(8) Suppose $(k, p) \neq (60, 31)$. Since $\hat{x}_i, \hat{x}_{28}^2, \hat{x}_{40}^3 \in I_7 + (\hat{x}_{40}^3)$ for $i = 4, 16, 24, 36$, we get $\lambda \neq 0$ by considering $\rho_1^*(\mathcal{P}^1 x_k) \pmod{I_7 + (\hat{x}_{40}^3)}$ as above.

Suppose next that $(k, p) = (60, 31)$. By a degree reason, we have

$$\rho_1^*(x_{60}) \equiv \alpha p_5^3 + \beta p_7 p_5 p_3 \pmod{I_8 + (\hat{x}_{40}^2)}$$

for $\alpha, \beta \in \mathbb{Z}/p$. Since $\hat{x}_i, \hat{x}_{40}^2 \in I_8 + (\hat{x}_{40}^2)$ for $i = 4, 16, 24, 36$ and $\rho_1^*(x_{48}) \equiv -200 p_7 p_5 \pmod{I_8}$, we have

$$\rho_1^*(\mathcal{P}^1 x_{48}) \equiv \mu \hat{x}_{48} \rho_1^*(x_{60}) \equiv -200 \mu (\alpha p_7 p_5^4 + \beta p_7^2 p_5^2 p_3) \pmod{I_8 + (\hat{x}_{40}^2)}$$

for some $\mu \in \mathbb{Z}/p$. By Proposition 2.12, we also have

$$\rho_1^*(\mathcal{P}^1 x_{48}) = \mathcal{P}^1 \rho_1^*(x_{48}) \equiv 10 p_7 p_5^4 + 11 p_7^2 p_5^2 p_3 \pmod{I_8 + (\hat{x}_{40}^2)}$$

Then we may put $(\alpha, \beta) = (17, 28)$ and $\mu = 1$. In the case (7), we have seen that $\mathcal{P}^1 x_{48} \equiv \mu x_{48} x_{60} \pmod{(x_{2i} \mid i \in \mathfrak{t}(E_8))^3}$, implying that $\mathcal{P}^1 \mathcal{P}^1 x_{48} \equiv (\lambda + 1) x_{48} x_{60}^2 \pmod{(x_{2i} \mid i \in \mathfrak{t}(E_8))^4}$, where $\mathcal{P}^1 x_{60} \equiv \lambda x_{60}^2 \pmod{(x_{2i} \mid i \in \mathfrak{t}(E_8))^3}$. Then for a degree reason, we get

$$\rho_1^*(\mathcal{P}^1 \mathcal{P}^1 x_{48}) \equiv (\lambda + 1) \hat{x}_{48} \rho_1^*(x_{60})^2 \equiv 21(\lambda + 1) p_7^3 p_5^3 p_3^2 \pmod{I_8 + (\hat{x}_{40}^2)}.$$

On the other hand, by the Adem relation $\mathcal{P}^1 \mathcal{P}^1 = 2\mathcal{P}^2$ and Proposition 2.12, we have

$$\rho_1^*(\mathcal{P}^1 \mathcal{P}^1 x_{48}) = \rho_1^*(2\mathcal{P}^2 x_{48}) = 2\mathcal{P}^2 \rho_1^*(x_{48}) \equiv 7 p_7^3 p_5^3 p_3^2 \pmod{I_8 + (\hat{x}_{40}^2)},$$

hence $\lambda \neq 0$.

3.2. The case of E_7 . Suppose that E_7 is p -regular, that is, $p > 18$. Then if $\mathcal{P}^1 x_k \pmod{(x_{2i} \mid i \in \mathfrak{t}(E_7))^3}$ is non-trivial, it is as in the following table.

| | $\mathcal{P}^1 x_k \pmod{(x_{2i} \mid i \in \mathfrak{t}(E_7))^3}$ | (k, p) |
|-----|---|--|
| (1) | $\lambda_1 x_4 x_{36} + \lambda_2 x_{12} x_{28} + \lambda_3 x_{16} x_{24} + \lambda_4 x_{20}^2$ | (4, 19) |
| (2) | $\lambda_1 x_{12} x_{36} + \lambda_2 x_{20} x_{28} + \lambda_3 x_{24}^2$ | (12, 19), (4, 23) |
| (3) | $\lambda_1 x_{16} x_{36} + \lambda_2 x_{24} x_{28}$ | (16, 19) |
| (4) | $\lambda_1 x_{20} x_{36} + \lambda_2 x_{28}^2$ | (20, 19), (12, 23) |
| (5) | $\lambda x_{24} x_{36}$ | (24, 19), (16, 23), (4, 29) |
| (6) | $\lambda x_{28} x_{36}$ | (28, 19), (4, 31) |
| (7) | λx_{36}^2 | (36, 19), (28, 23), (16, 29), (12, 31) |

(1) It is proved in [HK] that $\lambda_i \neq 0$ for $i = 1, 2, 3, 4$.

(2) Put $I = (p_1, p_3^2, c_6, \hat{x}_{16})$. Since $\hat{x}_4, \hat{x}_{12}^2, \hat{x}_{16} \in I$, by Corollary 2.7, we have

$$\rho_2^*(\mathcal{P}^1 x_k) \equiv \lambda_1 \hat{x}_{12} \hat{x}_{36} + \lambda_2 \hat{x}_{20} \hat{x}_{28} + \lambda_3 \hat{x}_{24}^2 \equiv 60\lambda_1 p_5 p_3 p_2^2 + 40\lambda_2 p_5^2 p_2 + \frac{25}{81} \lambda_3 p_2^6 \pmod{I}.$$

On the other hand, it follows from Proposition 2.13 that

$$\rho_2^*(\mathcal{P}^1 x_k) = \mathcal{P}^1 \rho_2(x_k) \equiv \begin{cases} 18p_5^2 p_2 + 10p_5 p_3 p_2^2 + p_2^6 & (p = 19) \\ 22p_5^2 p_2 + 7p_5 p_3 p_2^2 + 7p_2^6 & (p = 23) \end{cases} \pmod{I},$$

hence $\lambda_1 \neq 0$, $\lambda_2 \neq 0$ and $\lambda_3 \neq 0$.

(3) In this case, we have $(k, p) = (16, 19)$. Put $I = (p_1, p_3, c_6, \hat{x}_{16}^2)$. Since $\hat{x}_4, \hat{x}_{12}, \hat{x}_{16}^2 \in I$, it follows from Proposition 2.7 that

$$\rho_2^*(\mathcal{P}^1 x_{16}) \equiv \lambda_1 \hat{x}_{16} \hat{x}_{36} + \lambda_2 \hat{x}_{24} \hat{x}_{28} \equiv (13\lambda_1 + 9\lambda_2) p_5 p_4 p_2^2 + (9\lambda_1 + 14\lambda_2) p_5 p_2^4 \pmod{I}.$$

By Proposition 2.13, we also have $\rho_2^*(\mathcal{P}^1 x_{16}) = \mathcal{P}^1 \rho_2^*(x_{16}) \equiv 11p_5 p_4 p_2^2 + 14p_5 p_2^4 \pmod{I}$, implying $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$.

(4) Put $I = (p_1, p_3^2, c_6^2, \hat{x}_{12}, \hat{x}_{16}^2, \hat{x}_{24}, \hat{x}_{16} \hat{x}_{20}^2)$. Since $\hat{x}_i, \hat{x}_{16}^2, \hat{x}_{16} \hat{x}_{20}^2 \in I$ for $i = 4, 12, 24$, we have

$$\rho_2^*(\mathcal{P}^1 x_k) \equiv \lambda_1 \hat{x}_{20} \hat{x}_{36} + \lambda_2 \hat{x}_{28}^2 \equiv (-10\lambda_1 + 1600\lambda_2) p_5^2 p_2^2 + (\frac{2}{3}\lambda_1 - \frac{320}{3}\lambda_2) p_5 p_3 p_2^3 \pmod{I}.$$

By Proposition 2.13, we also have

$$\rho_2^*(\mathcal{P}^1 x_k) = \mathcal{P}^1 \rho_2^*(x_k) \equiv \begin{cases} 10p_5^2 p_2^2 + 12p_5 p_3 p_2^3 & (p = 19) \\ 15p_5^2 p_2^2 + 22p_5 p_3 p_2^3 & (p = 23) \end{cases} \pmod{I},$$

hence $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$.

(5) and (7) Put $I = (p_1, p_3, p_5^2, c_6, \hat{x}_{16})$ and $J = (p_1, p_3^2, c_6^2, \hat{x}_{12}, \hat{x}_{16}, \hat{x}_{20}^3, \hat{x}_{24}^2, \hat{x}_{20} \hat{x}_{24} \hat{x}_{28})$. Then since $\hat{x}_i, \hat{x}_{20}^2 \in I$ for $i = 4, 12, 16$ and $\hat{x}_i, \hat{x}_{20}^3, \hat{x}_{24}^2, \hat{x}_{20} \hat{x}_{24} \hat{x}_{28} \in J$ for $i = 4, 12, 16$, we have $\lambda \neq 0$ similarly to (4) of E_8 by considering $\rho_2^*(\mathcal{P}^1 x_k)$ modulo I and J respectively for (5) and (7).

(6) The case $p = 31$ follows from the above case of E_8 together with Corollary 2.7. Then we consider the case $p = 19$. Put $I = (p_1, p_3^2, c_6^2, \hat{x}_{12}, \hat{x}_{16}, \hat{x}_{20}^2, \hat{x}_{24}^2)$. Since $\hat{x}_i, \hat{x}_j^2 \in I$ for $i = 4, 12, 16$ and $j = 20, 24$, we get $\lambda \neq 0$ as above by considering $\rho_2^*(\mathcal{P}^1 x_k) \pmod{I}$.

3.3. The cases of E_6 and F_4 . We first consider the case of E_6 . Suppose that E_6 is p -regular, that is, $p \geq 13$. By an easy dimensional consideration, we see that if $\mathcal{P}^1 x_k \not\equiv 0 \pmod{(x_{2i} \mid i \in \mathfrak{t}(E_6))^3}$, it is as in the following table.

| | $\mathcal{P}^1 x_k \pmod{(x_{2i} \mid i \in \mathfrak{t}(E_6))^3}$ | (k, p) |
|-----|--|---|
| (1) | $\lambda_1 x_4 x_{24} + \lambda_2 x_{10} x_{18} + \lambda_3 x_{12} x_{16}$ | $(4, 13)$ |
| (2) | $\lambda_1 x_{10} x_{24} + \lambda_2 x_{16} x_{18}$ | $(10, 13)$ |
| (3) | $\lambda_1 x_{12} x_{24} + \lambda_2 x_{18}^2$ | $(12, 13), (4, 17)$ |
| (4) | $\lambda x_{16} x_{24}$ | $(16, 13), (4, 19)$ |
| (5) | $\lambda x_{18} x_{24}$ | $(18, 13), (10, 17)$ |
| (6) | λx_{24}^2 | $(24, 13), (16, 17), (12, 19), (4, 23)$ |

When $p = 19, 23$, the result follows from the above case of E_7 and Corollary 2.8.

(1) It is proved in [HK] that $\lambda_1 \neq 0$, $\lambda_2 \neq 0$ and $\lambda_3 \neq 0$.

(2) Put $I = (p_1, p_3^2, c_5^2)$. Since $\hat{x}_4, \hat{x}_{10}^2, \hat{x}_{12}^2 \in I$, we have

$$\rho_3^*(\mathcal{P}^1 x_{10}) \equiv \lambda_1 \hat{x}_{10} \hat{x}_{24} + \lambda_2 \hat{x}_{16} \hat{x}_{18} \equiv 5\lambda_1(-p_4 p_2 c_5 + \frac{1}{36} p_2^3 c_5) + \lambda_2(12p_4 p_2 c_5 + p_2^3 c_5) \pmod{I},$$

where $\hat{x}_{10} = c_5$ and $\hat{x}_{18} = p_2 c_5$. On the other hand, by Proposition 2.14, we have $\rho_3^*(\mathcal{P}^1 x_{10}) = \mathcal{P}^1 \rho_3^*(x_{10}) \equiv 6p_4 p_2 c_5 + 7p_2^3 c_5 \pmod{I}$ for $p = 13$, hence $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$.

(3) Put $I = (p_1, p_3^2, \hat{x}_{16})$. It is sufficient to consider the case $p = 13, 17$. Since $\hat{x}_i, \hat{x}_{12}^2 \in I$ for $i = 4, 16$,

$$\rho_3^*(\mathcal{P}^1 x_k) \equiv \lambda_1 \hat{x}_{12} \hat{x}_{24} + \lambda_2 \hat{x}_{18}^2 \equiv -\frac{10}{3} \lambda_1 p_3 p_2^3 + \lambda_2 p_2^2 c_5^2 \pmod{I}.$$

By Proposition 2.14, we have

$$\rho_3^*(\mathcal{P}^1 x_k) = \mathcal{P}^1 \rho_3^*(x_k) \equiv \begin{cases} 9p_3 p_2^3 + 5c_5^2 p_2^2 & (p = 13) \\ 13p_3 p_2^3 - 11c_5^2 p_2^2 & (p = 17) \end{cases} \pmod{I},$$

implying $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$.

(4), (5) and (6) Put $I = (p_1, p_3, p_4, c_5)$, $J = (p_1, p_3, c_5^2, \hat{x}_{16})$ and $K = (p_1, p_3, c_5, \hat{x}_{16})$. Then since $\hat{x}_i \in I$ for $i = 4, 10, 12$, $\hat{x}_i, \hat{x}_{10}^2 \in J$ for $i = 4, 12, 16$ and $\hat{x}_i \in K$ for $i = 4, 12, 10, 16$, we get $\lambda \neq 0$ similarly to (4) of E_8 by considering $\rho_3^*(\mathcal{P}^1 x_k)$ modulo I, J, K respectively for (4), (5) and (6).

We next consider the case of F_4 . Notice that F_4 is p -regular if and only if so is E_6 , and that as in the proof of Corollary 2.9, the map $\alpha_3^* : H^*(BE_6; \mathbb{Z}/p) \rightarrow H^*(BF_4; \mathbb{Z}/p)$ is surjective. Then the result for F_4 follows from that for E_6 above.

3.4. The case of G_2 . For a degree reason, if G_2 is p -regular and $\mathcal{P}^1 x_k \not\equiv 0 \pmod{(x_{2i} \mid i \in \mathfrak{t}(G_2))^3}$, then $(k, p) = (4, 7), (12, 7), (4, 11)$. Hence Theorem 2.2 for G_2 readily follows from Proposition 2.15.

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